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ON GAUGE FIXING AND QUANTIZATION OF CONSTRAINED HAMILTONIAN SYSTEMS

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ABSTRACT

In constrained Hamiltonian systems which possess first class constraints some subsidiary conditions should be imposed for detecting physical observables. This issue and quantization of the system are clarified. It is argued that the reduced phase space and Dirac method of quantization, generally, differ only in the definition of the Hilbert space one should use. For the dynamical systems possessing second class constraints the definition of physical Hilbert space in the BFV–BRST operator quantization method is different from the usual definition.

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Quantization of a classical Hamiltonian system can be achieved by the canonical quantization method [1]. If we ignore the ordering problems, it consists of replacing the classical Poisson brackets by quantum commutators when classically all the states on the phase space are accessible. This is no longer correct in the presence of constraints. An approach due to Dirac [2] is widely used for quantizing the constrained Hamiltonian systems [3], [4]. In spite of its wide use there is confusion in the application of it in the physics literature. Obtaining the Hamiltonian which should be used in gauge theories is not well-understood. This derives from the misunderstanding of the gauge fixing procedure. One of the aims of this work is to clarify this issue.

The constraints, which are present when there are some irrelevant phase space variables, can be classified as first and second class. Second class constraints are eliminated by altering some of the original Poisson bracket relations which is equivalent to imply the vanishing of the constraints as strong equations by using the Dirac procedure [2] or another method [5]. Vanishing of second class constraints strongly leads to a reduction in the number of the phase space variables. This reduced system can now be quantized by means of the canonical quantization method.

Presence of first class constraints change the scheme drastically. Unsimilar to the second class constraints they generate gauge transformations. A gauge fixing procedure is to introduce some subsidiary conditions as is done in the path integral approach [6]. At the classical level the constraints and the gauge fixing conditions can be imposed strongly to eliminate the unphysical variables. The resulting reduced phase space is built up by physical variables which now satisfy some new Poisson bracket relations (Dirac brackets). Thus one can apply the canonical quantization procedure. This is known as reduced phase space quantization[7]. It is obvious that treating the original first class constraints and the gauge fixing conditions as second class constraints[8] is equivalent to the reduced phase space method.

In Dirac method one applies the canonical quantization procedure before imposing the constraints and the first class constraints which become operators are used to define the physical subspace of the total Hilbert space. First class constraints are added to the canonical Hamiltonian by some Lagrange multipliers. This Hamiltonian can be used in the weak sense or in the path integral but generally cannot be used as an operator for defining

the time evolution of the physical states, by making use of the original Poisson brackets of the physical variables, except in the case where each one of the constraints is linear in a momentum or a coordinate variable.

The reduced phase space and Dirac method of quantization do not always lead to the same quantized system[9]. The latter seems to be the correct procedure. In the following we will show that if the physical subspace of the total Hilbert space is used as the Hilbert space of the reduced phase space quantization, it turns out to be equivalent to the Dirac method i.e. only the physical observables are quantized by replacing their Poisson brackets with the quantum commutators, but they act on the physical Hilbert space. Thus when these two spaces are isomorphic, they should lead to the same quantized system[1].

Recently in [11] canonical quantization in the presence of gauge invariances is studied by stressing the effects of factor ordering. In the second paper of [11] it is also shown that even the Hamiltonians can be different in the reduced phase space and Dirac quantization methods, the physical states evolved by these Hamiltonians remain in the physical subspace.

When the dynamical system possesses some second class constraints there exists another method given by Batalin and Fradkin [12] :BFV-BRST operator quantization method (for applications of the method see [13]). One enlarges the phase space such that the original second class constraints become converted into the first class ones, so that the number of the physical degrees of freedom remains unaltered. We will show that these first class constraints should be handled differently from the usual ones when one uses them to define the physical Hilbert space for not altering the original one.

Let us consider a bosonic system[2] whose dynamics can be derived from a singular Lagrangian $\mathcal{L} = \mathcal{L}(q_\mu, \dot{q}_\mu)$; $\mu = 1, \ldots, N$, which leads to consistent equations of motion. Definition of canonical momenta, $p_\mu = \partial \mathcal{L}/\partial \dot{q}_\mu$, will lead to some relations between phase space variables $\varphi_k(q_\mu, p_\mu) = 0$; $1, \ldots, M < N$. These primary constraints can be incorporated in the

[1] This is discrepant with the results obtained in [10]. In fact the difference of the hamiltonians given in [10] is due to ordering properties. It is not due to different quantisation schemes.

[2] In the case of fermionic variables one only needs to replace the Poisson brackets with the generalized Poisson brackets, in the following.

Hamiltonian as

$$H_1 = H_c + u_k \varphi_k,$$

where H_c is the canonical Hamiltonian and u_k are yet unspecified coefficents. By defining the usual Poisson bracket relations between the conjugate variables the time evolution of constraints can be obtained as their Poisson brackets with the Hamiltonian. They should be invariant under time evolution. This may lead to some new constraints or/and specifying of some u_k. Invariance of these new constraints in time may lead some other constraints and/or specifying of some other coefficents and so on (all the constraints which are not primary will be called secondary constraints). Let us denote all of the constraints which are linearly independent as Υ_K ; $K = 1, \ldots, M_1 < N$. Some of these constraints may yield nonzero Poisson brackets with some of the others on the constraint surface. These are called second class constraints (the number of them is always even) and can be eliminated by making use of Dirac brackets or by means of the procedure given in [5], which is equivalent to using the Dirac brackets [14]. By these new brackets, vanishing of second class constraints is imposed as strong equations. Thus if there are $2M_2 < M_1$ second class constraints by vanishing of them strongly one removes M_2 of coordinates and their conjugate momenta from the phase space. The application of these strong conditions may change the canonical part of the Hamiltonian so that the new Hamiltonian reads

$$H = H_0(q_a, p_a) + v_\alpha \phi_\alpha(q_a, p_a) \quad a = 1, \ldots, n; \alpha = 1 \ldots, m < n, \qquad (1)$$

where ϕ_α are first class (primary or secondary) constraints.

First class constraints generate gauge invariances. Hence a gauge fixing procedure is needed for detecting the physical observables. A gauge fixing procedure is to introduce some subsidary conditions $\chi_\alpha = 0$, which are supposed to have a vanishing Poisson bracket with H_0 and satisfy

$$det\ \{\phi_\alpha, \chi_\beta\}|_\Sigma \neq 0,$$

where Σ denotes the surface defined by the constraints and the gauge fixing conditions. This a physical gauge where all gauge freedom is removed. χ_α cannot be regarded as the usual subsidiary conditions [2], which one imposes on a Hamiltonian system for obtaining a different one. They designate the

chosen gauge and the physical amplitudes should be independent of gauge fixing conditions. Thus one cannot expect that they lead to specifying of some coefficients in (1) as the normal subsidiary conditions. In fact the invariance of them under time evolution leads to the vanishing of all v_a coefficents. The Hamiltonian (1) is not suitable to be used in this physical gauge, except the case which we will see below. It is the Hamiltonian in the weak sense. Of course in the path integral approach [6], where a first order Lagrangian is utilized, the Dirac Hamiltonian (1) can be used in the physical gauge.

For detecting the suitable Hamiltonian in the reduced phase space, which is obtained by imposing the vanishing of the constraints and the gauge fixing conditions strongly, one needs to use the first order Lagrangian

$$L = \int dt\ (p_a \dot{q}_a - H), \tag{2}$$

where H is given in (1). In the case where each one of the constraints is linear in a momentum or a coordinate the first order Lagrangian in the reduced phase space reads

$$L^* = \int dt\ (p_i \dot{q}_i - H^*)\ ;\ i = 1, \ldots, n-m, \tag{3}$$

where

$$H^* = H_0|_{\chi_a = 0}.$$

This is the exception which is announced above: when each one of the constraints is linear in a variable, the Dirac Hamiltonian (1) can be used in the physical gauge, without altering the original Poisson bracket relations of the physical variables.

In principle the constraints which are non-linear in variables can be transformed into momenta by means of a local canonical transformation. Usually due to the non-linear nature of these transformations operator ordering can no longer be ignored. The quantized system changes drastically as we will display with an example. Hence we will not perform any canonical transfo...ation, so that non-... haracter of the constraints make sense. By keeping the original variables, when each constraint doesn't depend on only one variable linearly, the reduced phase space Lagrangian reads

$$L^* = \int dt\ (p'_i \dot{q}'_i - \tilde{H}^*), \tag{4}$$

where primed variables may be equal to the unprimed ones or obtained from them by rescaling (we suppose that gauge fixing conditions are chosen such that the Dirac brackets of the unprimed variables are different from the original brackets up to a constant). Now $\tilde{H}^*$ can not be obtained from the Dirac Hamiltonian (1) and as we will see it depends on gauge fixing conditions. Hence we see that in the physical gauges the reduced phase space Hamiltonian should be obtained from the first order Lagrangian not from the Dirac Hamiltonian.

As an example we can take the systems which possess reparametrization invariance. In this case the canonical Hamiltonian vanishes $H_0 = 0$. For example in $d = 4$ relativistic massive point particle case dynamics depends only on a first class constraint $\phi = p_\mu^2 - m^2$. One may fix the parametrization time as $\chi = x_0 - t = 0$ [3] so that $\dot{x}_0 = 1$. Thus by solving $\phi = 0$ for p_0 ($p_0 > 0$) one obtains

$$L^* = \int dt (p_i \dot{x}_i - \sqrt{p_i^2 + m^2}) \tag{5}$$

where i runs over the space indicies. Thus the Hamiltonian in this physical gauge is $\sqrt{p_i^2 + m^2}$.

In most of the interesting cases working in a positive definite Hilbert space is not desired for keeping the manifest covariance of the system. In this case the Dirac Hamiltonian can be used in the weak sense and a gauge fixing is not performed by introducing some subsidiary conditions but by specifying some of the v_a coefficents. In the relativistic point particle case one can choose $v = 1$ so that the Hamiltonian reads $H = p^2 + m^2$. In the bosonic string case one can fix the gauge by choosing a common time for the string. In this physical gauge (light-cone) one has to use the above procedure for detecting the Hamiltonian[3] which gives the evolution in the common time. Instead in the Dirac approach one fixes the v_a coefficents such that the Hamiltonian which follows from the Dirac Hamiltonian is $H = L_0$ [15].

Till now we have discussed the constrained Hamiltonian systems at the classical level. Quantization can be achieved by mean of path integral or operator methods. We will only deal with the latter.

In the reduced phase space method the physical variables which are detected in a physical gauge are quantized by replacing the Poisson bracket

relations (Dirac brackets) with the commutators. Thus the Hilbert space depends only on the physical coordinates. In the case where only the second class constraints are present the Dirac procedure is equivalent to the former one because in each of these procedures one implies the vanishing of the constraints as strong equations. But when there are first class constraints, in the Dirac approach, one does not imply any subsidiary condition as gauge fixing. One replaces all the Poisson brackets with commutators and works out the realization of momenta in terms of coordinates. Thus the constraints become operators and can be used to define the physical subspace of the total Hilbert space as

$$\phi_\alpha \Psi_{phys.} = 0. \tag{6}$$

Dynamics of the relativistic point particle can be explored in the relativistic gauge $v = 1$. In this gauge the Dirac Hamiltonian vanishes on the physical states. It only yields the definition of physical states as

$$(\frac{\partial^2}{\partial x_\mu^2} + m^2)\Psi_{phys.} = 0,$$

which is the Klein-Gordon equation.

As it is clarified above the Dirac Hamiltonian cannot be used to evolve the physical Hilbert states except in the case where each of the constraints is linear in a momentum or a coordinate operator. Thus for detecting the suitable Hamiltonian operator which acts on the physical states one has to perform a gauge fixing like in the reduced phase space method at the classical level. But the definition of the physical states should still remain as in (6).

We know that the Hilbert space of the reduced phase space is not always isomorphic to the physical subspace of the total Hilbert space of the Dirac approach. This derives from the fact that the Hilbert space of the reduced phase space can heuristicaly be visualized as the subspace of the total Hilbert space of the Dirac quantization which is defined as to be annihilated by constraints and also by gauge fixing conditions. Thus if one wants to evolve the elements of the physical Hilbert space he needs to detect the reduced phase space Hamiltonian without altering the definition of the physical Hilbert space. Thus the unique difference between the mentioned

operator quantization methods is the utilized Hilbert spaces. When these two spaces are isomorphic they all lead to the same quantum system.

We would like to exhibit the above discussion in an example. The following Lagrangian $(x, y \in \mathbf{R}^2/\{0,0\})$

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \lambda(x\dot{y} - y\dot{x}) + \frac{1}{2}\lambda^2(x^2 + y^2) - V(x^2 + y^2), \tag{7}$$

which is studied in [16], [17] is singular.. The definition of canonical momenta leads to the canonical Hamiltonian:

$$H_0 = \frac{1}{2}(p_x^2 + p_y^2) + V(x^2 + y^2), \tag{8}$$

and two independent first class constraints:

$$\begin{aligned} \phi_1 &= p_\lambda, \\ \phi_2 &= xp_y - yp_x. \end{aligned} \tag{9}$$

The following subsidiary conditions

$$\begin{aligned} \chi_1 &= \lambda, \\ \chi_2 &= x - y, \end{aligned} \tag{10}$$

can be used as physical gauge fixing conditions, because

$$det\ \{\phi_\alpha, \chi_\beta\} = (x^2 + y^2).$$

When the constraints and gauge fixing conditions are used, the first order Lagrangian reads

$$L^* = \int dt[p_x' \dot{x}' - \frac{1}{2}p_x'^2 - V'(x')], \tag{11}$$

where $p_x' = \sqrt{2}p_x\ x' = \sqrt{2}x$. The Poisson bracket relation between p and x should be found by taking into account the contraints and the gauge fixing conditions (Dirac brackets). So that the rescaled observables satisfy the following Poisson bracket relation

$$\{p_x', x'\} = 1. \tag{12}$$

(11) exposes the Hamiltonian which should be used in the gauge (10) as

$$H^* = \frac{1}{2}p_x'^2 + V'(x').$$

Now quantization can be performed as usual by realizing the quantum commutator obtained from (12) as $p_x' = -id/dx'$. Even the physical operators depend only on x' they do not act on the Hilbert space states which depend only on x'. They act on the states of physical subspace defined as in (6). The physical states are

$$\Psi_{phys.} = \Psi_{phys.}(r) \; ; \; r = (x^2 + y^2). \tag{13}$$

Thus the momentum dependent part of the Hamiltonian leads to

$$\frac{d^2}{dx^2}\Psi_{phys.}(r) = [\frac{1}{2r}\frac{d}{dr} + \frac{d^2}{dr^2}]\Psi_{phys.}(r). \tag{14}$$

This is consistent with the results of [16]. Of course if one uses the states which depend only x in the Hamiltonian (14), the first term on the right hand side will be absent. Thus the reduced phase space and Dirac quantization methods do not lead to the same quantum theory[3].

The dynamical system outlined by (8), (9) can also be studied in the polar coordinates in (x, y) space. The second constraint in (9) reads $\phi_2 = p_\theta$ which leads, by making use of the other constraint, to the physical states which depend on r only. The reduced phase space hamiltonian in terms of polar coordinates is $H^* = \frac{1}{2}p_r^2 + V(r)$. By making use of the usual realization of the quantum commutators the momentum dependent part of Hamiltonian yields

$$\frac{d^2}{dr^2}\Psi_{phys.}(r). \tag{15}$$

In this case there is no difference between the two different quantization schemes which are discussed. Each of the constraints is just one of the momenta so ie physical subspace of the total Hilbert space of Dirac quantization method is isomorphic to the Hilbert space of reduced phase space method. The difference between (14) and (15) derives from the fact

[3] This result is obtained also in [18]. But in that paper the Dirac Hamiltonian is used in the strong sense so that their conclusions are not justified.

that when a non-linear canonical transformation is performed operator ordering should somehow taken into account [17]. Hence the difference of (14) and (15) is not due to the different quantization approachs but due to using Cartesian and polar coordinates in the original Lagrangian. If the Cartesian coordinates are taken to be privileged, one has to define p_r consistently[4].

In the BFV-BRST operator quantization method the physical subspace should be defined as above in the case when there are only first class constraints. But in this formalism when there are second class constraints one converts them into first class ones by enlarging the phase space. The first class constraints with gauge fixing conditions are visulized as second class constraints. Thus, in this case, in order to respect the original system the physical states should be defined as in the reduced phase space method.

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[4] The difference between the Hamiltonians which is found in [10] is the difference between (14) and (15) (in the language of that article $\lambda = r^2$). In the examples of [10] the privileged coordinates are the polar ones, so that there is no difference between the two different quantisation methods

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